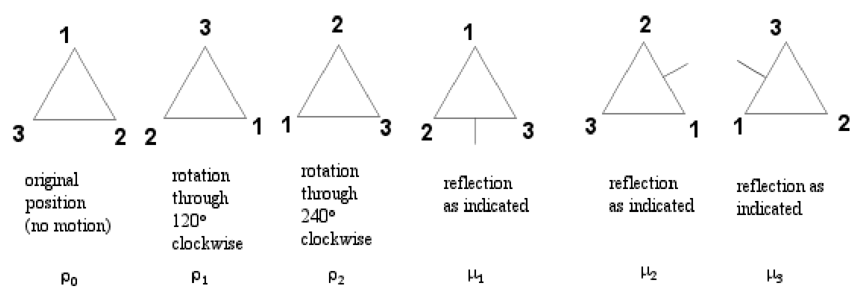
Introductory worksheet: some useful examples

1. **Symmetries.**
   We can think of a symmetry of a plane figure as a rigid motion of the figure that results in the figure simply being repositioned on top of its original outline. (We can extend this idea to symmetries of figures in three dimensions.)

   It is the final position of the figure that is important, not the motion as such. For example, if we rotate the triangle below through 120° or 480° the triangle ends up in the same final position, so we do not think of these as distinct symmetries.

   (a) Consider the symmetries of an equilateral triangle, as illustrated below. Each sketch shows the resulting position when the specified motion is applied to the triangle starting in the original position).



   Draw an equilateral triangle on a sheet of paper, and number the vertices 1, 2, and 3 as shown in the sketch. Now cut the triangle out, leaving the hole intact. This cut-out triangle will be helpful as you work through what follows.

   We can apply one of the rigid motions, and then, *continuing from the new position of the triangle*, apply another of the rigid motions. We can then record the overall effect as one of six symmetries listed above.

   For example if we first apply μ_1, then (continuing from the resulting position) apply ρ_1 (a 120° clockwise rotation) we end up with μ_2. Using our usual function composition notation, we can think of this as $\rho_1 \circ \mu_1$. In fact we write $\rho_1 \circ \mu_1 = \mu_2$.

It is useful to write the results of all such combinations in table form, as shown below. We show the result of $\rho_1 \circ \mu_1$ in the row labelled ρ_1 and the column labelled μ_1.

Important: the convention is that we enter the result of $\rho_1 \circ \mu_1$ into the *row* corresponding to ρ_1 and the *column* corresponding to μ_1, even though when we do these motions we *first* do the reflection μ_1 *and then* the rotation ρ_1.

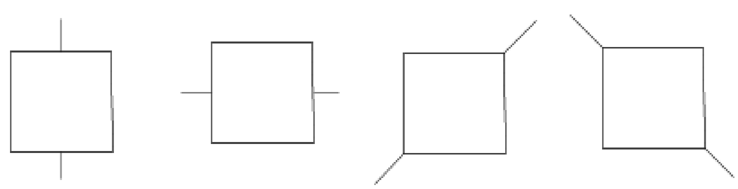
$\circ$	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0						
ρ_1				μ_2		
ρ_2						
μ_1						
μ_2						
μ_3						

Use the cut-out triangle to determine the compositions and use this to complete the table.
We will denote this collection of symmetries, together with composition, by D_3.

(b) Develop a similar table for the symmetries of a square. We will use the notation D_4 for the symmetries of a square together with composition.

So that we are all using the same notation, call the original position ρ_0, denote clockwise rotations through 90° , 180°, and 270° by ρ_1, ρ_2 and ρ_3 respectively, and label the reflections as indicated in the sketch.

(Start by sketching and naming all eight symmetries of the square for reference, and as for the triangle, use a cut-out square to calculate the compositions.)



μ_1 μ_2 δ_1 δ_2

2. **Clock arithmetic**

   Consider the numbers on a clock, and imagine 0 in place of 12. We will denote this set by $\mathbb{Z}_{12}$). So $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

   We define "addition modulo 12" on this set as follows: for a and b in this set, $a + b \pmod{12}$ is the hour on the clock-face that is b hours after a. For example, $9 + 5 = 2 \pmod{12}$ (since 0 is 3 hours after 9, and we need an additional 2 hours after that.)
   We can also define "multiplication mod 12" on $\mathbb{Z}_{12}$ by thinking of this as repeated addition modulo 12. So for a and b in the $\mathbb{Z}_{12}$, we think of $ab \pmod{12}$ as the result of adding b to itself a times, modulo 12.

   For example $(3)(7) = 9 \pmod{12}$.

   There is nothing special about 12 here; we can just as easily define addition and multiplication mod n on the set $\{0, 1, 2, \ldots, n-1\}$ for any fixed positive integer n. Simply imagine a clock-face with the numbers $\{0, 1, 2, \ldots, n-1\}$ in place of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, and for a and b in this set, define $a + b \pmod{n}$ to be the hour on this clock-face that is b hours after a . Define $ab \pmod{n}$ to be the result of adding b to itself a times, modulo n.

   We can draw up tables for these operations, just as we did for symmetries.

   (a) Consider $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. Draw up a table for addition mod 5, and a separate table for multiplication mod 5.

   (b) Now consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Draw up a table for addition mod 6, and a separate table for multiplication mod 6.