Math 302 - Solutions to Exam 1

1. Suppose G is a group that has exactly eight elements of order 3. How many subgroups of order 3 does G have?

   Solution: Let a be one of the elements of order three in G. Then $< a >= \{a, a^2, e\}$ is one subgroup of order three. Note that $|a^2| = 3$, so we've now taken account of two of the eight. In this manner we can see that G has **4** subgroups of order 3.

2. An Abelian group of order 6 must be cyclic.

   Proof: Let G be an Abelian group of order 6, and let $a \in G$. Note that $|a| \leq 6$ (otherwise G would get too big.

   - If $|a| = 6$, then $G =< a >$, and we're done.
   - If $|a| = 5$, then there is a nonidentity element $b \in G$, $b \notin < a >$. But then $e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b$, and a^4b are all distinct elements of G, contradicting $|G| = 6$. So a cannot have order 5.
   - If $|a| = 4$, we reach a similar contradiction, so a cannot have order 4.
   - if $|a| = 3$, then there is a nonidentity element $b \in G$, $b \notin < a >$. If $|b| > 2$, then again we get too many distinct elements in G $(e, a, a^2, b, ab, a^2b, b^2, ...)$, so it must be the case that $|b| = 2$; i.e., $b^2 = e$. We get $< ab >= \{ab, a^2b^2 = a^2, a^3b = b, ab^2 = a, a^2b, a^3b^2 = e\}$, with six distinct elements, so G is cyclic.
   - Finally, if $|a| = 2$, then choose a nonidentity element $b \neq a$. If $|b| = 2$, then if we try to build G we have, so far, $\{e, a, b, ab\}$, which is a subgroup, so there must be an element c, of order at least two, not in that subgroup. But then closure would force us to have at least 8 distinct elements $(e, a, b, ab, c, ac, bc, abc, ...)$, again too big. If $|b| > 3$, again the group gets too big. If $|b| = 3$, then $|ab| = 6$, and G must be cyclic. □

3. The group $\mathbb{Z}_5{}^* \times \mathbb{Z}_2$ is not cyclic, as no single element can generate the whole group.

4. Let G be a group and let H be a subgroup of G. Define $N(H) = \{x \in G | xHx^{-1} = H\}$. Claim: $N(H)$ (the *normalizer* of H) is a subgroup of G.

   Proof: We'll use the two-step subgroup test.
   (i) Let $x, y \in N(H)$. Then $xHx^{-1} = H$ and $yHy^{-1} = H$. (NOTE: these equalities are about sets. We can conclude that, for any element h in H, xhx^{-1} is also an element of H, but it is not necessarily equal to h.) Taking the second equation and multiplying on the left by x and on the right by x^{-1}, we have

$xyHy^{-1}x^{-1} = xHx^{-1}$. Using the first equation, and noting that $(xy)^{-1} = y^{-1}x^{-1}$, we have $xyH(xy)^{-1} = H$, and hence $xy \in N(H)$.

(ii) Let $x \in N(H)$.Then $xHx^{-1} = H$. Multiplying on the left by x^{-1} and on the right by x, we see
$H = x^{-1}Hx = x^{-1}H(x^{-1})^{-1}$. Therefore $x^{-1} \in N(H)$.

Thus $N(H)$ is a subgroup of G. □

5. Let G be a group, and let S be a nonempty subset of G. The set $< S >$ is a subgroup of G.

   Proof: We apply the two-step subgroup test.

   (a) Let $a, b \in < S >$. By definition, a and b are finite products of elements of S and their inverses, and hence ab is such a finite product. Since $< S >$ contains all possible such products, $ab \in < S >$.

   (b) For any $a \in < S >$, $a = a_1a_2...a_k$, where the a_i are elements S or inverses of elements of S. Now $a^{-1} = a_k^{-1}...a_2^{-1}a_1^{-1}$, also a product of elements of S or their inverses, and hence $a^{-1} \in < S >$.

   □

6. Find two elements of D_4 (the group of symmetries of the square) that generate the whole group. Then do the same for H, the group defined in Problem 7 of Problem Set 3.

   Solution: The set $\{\rho_1, \delta_1\}$ will generate D_4. (This solution is not unique.)
   The set $\{I, J\}$ will generate H. (Again, this solution is not unique.).

7. The set $M = \{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in \mathbb{R}\}$ is a subgroup of $SL_2(\mathbb{R})$.

   Proof: Let $A, B \in M$. Then $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$ for some $\alpha, \beta \in \mathbb{R}$. Hence $AB = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$.
   Using trigonometric identities, we see that $AB = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$.
   Because $\alpha + \beta$ is a real number, we see that $AB \in M$, and hence M is closed under the operation of $SL_2(\mathbb{R})$.
   Also, with A as above, note that $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}$,

and thus $A^{-1} \in M$.
Therefore, by the two-step subgroup test, M is a subgroup of $SL_2(\mathbb{R})$.. $\square$

8. Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that the set $C = \{E, R, R^2, R^3, D, RD, R^2D, R^3D\}$ is a subgroup of $GL_2(\mathbb{R}\}$. Then compare it to other groups of order eight that you've seen. What do you notice?

Solution: We construct the multiplication table for this set:

	E	R	R^2	R^3	D	RD	R^2D	R^3D
E	E	R	R^2	R^3	D	RD	R^2D	R^3D
R	R	R^2	R^3	E	RD	R^2D	R^3D	D
R^2	R^2	R^3	E	R	R^2D	R^3D	D	RD
R^3	R^3	E	R	R^2	R^3D	D	RD	R^2D
D	D	R^3D	R^2D	RD	E	R^3	R^2	R
RD	RD	D	R^3D	R^2D	R	E	R^3	R^2
R^2D	R^2D	RD	D	R^3D	R^2	R	E	R^3
R^3D	R^3D	R^2D	RD	D	R^3	R^2	R	E

Note that the set is closed under matrix multiplication, and that E, the identity matrix, appears in every row and column, so the set is also closed under taking inverses. Therefore it is a subgroup of $GL_2(\mathbb{R}\}$. $\square$

This group, unlike $\mathbb{Z}_8$ under addition or the group in problem 3, is not abelian. Like both D_4 and the group H from Problem Set 3, this group can be generated with two elements (in this case, D and R). Upon closer inspection, we notice that this group is more like D_4, because it has six five elements of order two.
In fact, if we regard each of these matrices as representing a linear transformation of $\mathbb{R}^2$, notice that R represents a 90-degree rotation around the origin, and D represents a reflection across the line $y = x$. (Compute $R\vec{v}$ and $D\vec{v}$ for a few vectors to see this in action.) So the connection of C with D_4 is quite close; they're the same group, in some sense.
While we're on the subject of linear transformations of $\mathbb{R}^2$, the matrices in problem 7 represent all of the rotations around the origin (but no reflections).