Abstract Algebra – Worksheet 6

Definition: Suppose that G and H are groups. Define an operation on $G \times H$ as follows: for all (g_1, h_1) and (g_2, h_2) in $G \times H$, define $(g_1, h_1)(g_2, h_2)$ to be $(g_1 g_2, h_1 h_2)$, where the operation in the first coordinate is the operation in G, and the operation in the second coordinate is the operation in H. (We often express this idea by saying that the operation is defined "component-wise".)

1. Determine the operation table for $\mathbb{Z}_2 \times \mathbb{Z}_3$, where the operation is defined component-wise. (Use additive notation, since the operation is based on addition.)
2. Prove that if G and H are groups, then $G \times H$, with operation defined component-wise, is a group. (Use multiplicative notation, since there is nothing to indicate that additive notation is appropriate).
3. Prove or disprove: if G and H are abelian groups, then $G \times H$ is abelian.
4. Consider the group ${\mathbb{Z}_5}^* \times \mathbb{Z}_2$ (operation in the group on the left is multiplication mod 5, and the operation in the group on the right is addition mod 2.) Draw up the operation table for this group.
5. **Definition:** Suppose G is a group with $a \in G$. We let $\langle a \rangle$ denote the set $\{a^n | n \in \mathbb{Z}\}$ together with the operation inherited from G.
   (a) Find $< 2 >$ in $\mathbb{Z}_8$.
   (b) Find $< 2 >$ in $\mathbb{Z}_9$.
   (c) FInd $< 2 >$ in $\mathbb{Z}$.
   (d) Prove that for any group G, and any $a \in G$, $< a >$ is a subgroup of G.