Abstract Algebra – Worksheet 4

You have shown in problem 12 that $(a^n)^{-1}$ and $(a^{-1})^n$ have unambiguous meanings, and are in fact equal. The symbol a^{-n}, on the other hand, is not automatically defined by the definitions already given. It is convenient to define a^{-n} as simply another notation for $(a^n)^{-1}$ and $(a^{-1})^n$:

Definition: In a group G with $a \in G$, we define a^{-n} to be $(a^n)^{-1}$. Also we define a^0 to be the identity, e.

As we said, we cannot simply assume that exponents will have the same properties in an arbitrary group as they do when working with real numbers. Some familiar properties of exponents for real numbers are in fact false in certain groups. The next two problems establish two basic principles that *do* apply in an arbitrary group.

14. Suppose G is a group, with $a \in G$. Prove that $a^m a^n = a^{m+n}$ for all integers m and n.

15. Suppose G is a group, with $a \in G$. Prove that $(a^m)^n = a^{mn}$ for all integers m and n.

16. Suppose G is a group,with a, b and x in G. If $x = a^{-1}b$, can we conclude that $xa = b$? Either prove this conclusion true, or provide a counterexample.

17. Suppose G is a group, with a and b in G. Prove that if $ab = e$, then $ba = e$. Use this to prove that if G is a group, with a and b in G and $ab = e$, then a is the inverse of b.

18. Prove or disprove, as appropriate: In a group, inverses are unique. (More precisely: if G is a group, and if $a \in G$, then there is a unique inverse for a in G.)

19. Prove or disprove, as appropriate: Suppose G is a group, with a, b and c in G. If $ac = bc$, then $a = b$.

20. Prove or disprove, as appropriate: If G is a group, with a and b in G, then $(ab)^2 = a^2b^2$

21. Prove or disprove, as appropriate: If G is a group, with a and b in G, then $(ab)^{-1} = a^{-1}b^{-1}$.

22. Prove or disprove, as appropriate: If G is a group, with a and b in G, then $(ab)^{-1} = b^{-1}a^{-1}$.

Historically, the central focus of abstract algebra was the solution of equations. The following problem gives an indication of the connection:

23. Suppose G is group, with a and b in G. Consider the equation $ax = b$.

   (a) Prove that $a^{-1}b \in G$.
   (b) Prove by substituting that $x = a^{-1}b$ is a solution for the equation.
   (c) Prove that $x = a^{-1}b$ is the *only* solution for the equation $ax = b$; that is, this solution is *unique*.

You have thus shown that if G is a group, then for all a and b in G, there is a *unique* solution in G for the equation $ax = b$. Similarly there is a unique solution in G for $xa = b$.