Abstract Algebra – Worksheet 2

Notational conventions:
$\mathbb{N}$: The set of natural numbers:$\{1, 2, 3 \ldots\}$
$\mathbb{Z}$: The set of integers: $\{\ldots - 2, -1, 0, 1, 2 \ldots\}$
$\mathbb{Q}$: The set of rational numbers: $\{\frac{n}{m} | n, m \in \mathbb{Z}, m \neq 0\}$
$\mathbb{R}$: The set of real numbers.
$\mathbb{C}$: The set of complex numbers: $\{a + bi \mid a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$.

Definition: A *binary operation* $*$ on a set A is a function which sends $A \times A$ to A. We denote the image of (x, y) under this function by $x * y$.

1. (ex) Which of the following are binary operations on the specified set? If not, explain why not.
   (a) Addition on $\mathbb{Z}$.
   (b) Subtraction on $\mathbb{N}$.
   (c) Division on $\mathbb{R}$.
   (d) Division on $\mathbb{Z}\backslash\{0\}$.
   (e) Composition on D_4.
   (f) Composition on the set of rotations in D_4.
   (g) Multiplication mod 6 on $\mathbb{Z}_6$.

Definitions:
• A binary operation $*$ on a set A is said to be *associative* if $(a * b) * c = a * (b * c)$ for all a, b and c in A.
• A binary operation $*$ on A is said to be *commutative* if $a * b = b * a$ for all a and b in A.
• Suppose $*$ is a binary operation on A. An element e of A is said to be an *identity* for $*$ if $a * e = e * a = a$ for all $a \in A$.
• Suppose $*$ is a binary operation on A, with identity element e. Let $a \in A$. An element b of A is said to be an *inverse* of a with respect to $*$ if $a * b = b * a = e$.

2. (ex) Refer back to those operations in question 1 that *were* binary operations. Do the following for each of these binary operations.
   (a) Determine whether the operation is associative, and if not, prove that the operation is not associative.
   (b) State whether there is an identity for the operation, and if so, identify it.
   (c) If there is an identity for the operation, determine which elements (if any) have inverses.

3. (ex) Determine which of the binary operations in question 1 are commutative, and if not, provide proof that the operation is not commutative.

4. Prove that if a binary operation $*$ on a set A has an identity element, then that identity element is unique.

Definition: A *group* is a set G together with a binary operation $*$ on G satisfying the following:

(a) The operation $*$ is associative.

(b) There is an element in G which is an identity for $*$.

(c) Every element in G has an inverse with respect to $*$ in G.

We denote the group by $\langle G, * \rangle$.

We refer to the set G as the *underlying set* of the group $\langle G, * \rangle$. (However if the specific operation is clear from the context, or is not important in the context, we sometimes simply write G instead of $\langle G, * \rangle$ for the group, and speak of "the group G".)

Note that we assume all the familiar properties of the operations on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, and you may use these freely in all that follows.

5. (ex) Which of the following are groups? If not, explain why not.
   (a) $\langle \mathbb{Z}, + \rangle$
   (b) $\langle \mathbb{Z}, - \rangle$
   (c) $\langle \mathbb{Z}, \times \rangle$
   (d) $\langle \mathbb{Z}, \div \rangle$
   (e) $\langle \mathbb{R}^+, \times \rangle$ ($\mathbb{R}^+$ denotes the set of positive real numbers.)
   (f) The set of symmetries of a regular pentagon with operation composition.
   (g) $\mathbb{Z}_6$ with operation addition mod 6.
   (h) $\mathbb{Z}_6$ with operation multiplication mod 6.
   (i) $\mathbb{Z}_6 \setminus \{0\}$ with operation multiplication mod 6.
   (j) $\mathbb{Z}_5 \setminus \{0\}$ with operation multiplication mod 5.

6. Prove that the following is, or is not a group, as appropriate.
   The set $S = \mathbb{R} \setminus \{1\}$ with operation defined by $a * b = a + b - ab$ for all a and b in S. (On the right side of the equation, the operations are the usual addition and multiplication in $\mathbb{R}$.)

7. Prove that the following is, or is not a group, as appropriate.
   The set $M_2(\mathbb{R})$ of all 2 by 2 matrices, with real numbers as entries, and operation matrix multiplication.