Math 302 – Abstract Algebra

Problem Set 9

Due Thursday, May 5

Definition. A *ring* R is a set with two binary operations, addition and multiplication (denoted as you'd expect) such that for all $a, b, c \in R$,

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There is an additive identity 0.
4. There is an element $-a \in R$ such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$.
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Examples

- The set $\mathbb{Z}$ of integers under ordinary addition and multiplication.
- The set $2\mathbb{Z}$ of even integers under ordinary addition and multiplication.
- The set $\mathbb{Z}_n = \{0, 1, ..., n - 1\}$ under addition and multiplication modulo n.
- The set $\mathbb{R}$ of real numbers under ordinary addition and multiplication.
- The set $M_2(\mathbb{Z})$ of 2×2 matrices with integer entries.
- The set $M_2(\mathbb{R})$ of 2×2 matrices with real number entries.

Exercises:

1. (a) A ring is *commutative* if its multiplication is a commutative operation. Which of the above rings is commutative?

   (b) Which of the above rings have multiplicative identities? (A multiplicative identity is called a *unity* of the ring.)

   (c) A nonzero element of a ring with unity is called a *unit* if it has a multiplicative inverse. Describe the units in each of the rings with identity.

2. Let R be a ring, and let $a, b, c \in R$. Prove that

   (a) $a0 = 0a = 0$.

   (b) $a(-b) = (-a)b = -(ab)$.

   (c) $(-a)(-b) = ab$.

3. Show, by example, that for fixed nonzero elements a and b in a ring R, the equation $ax = b$ can have more than one solution.

4. Let R be a commutative ring with unity, and let $U(R)$ be the set of units of R. Prove that $U(R)$ is a group under the multiplication of R.

5. The set of *Gaussian Integers* is $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$; it is a ring under ordinary addition and multiplication of complex numbers. Find $U(\mathbb{Z}[i])$.

6. A subset S of a ring R is a *subring of* R if S is itself a ring with the operations of R. Describe the subrings of $\mathbb{Z}$.

7. Show that a ring that is cyclic under addition is commutative.

8. Let R be a ring. Prove that $a^2 - b^2 = (a + b)(a - b)$ for all $a, b \in R$ if and only if R is commutative.