Math 302 – Abstract Algebra

Problem Set 3

Due Thursday, March 10

Definition: Let G be a group, and let H be a subset of G. We say that H is a *subgroup* of G if H is a group in its own right under the operation of G.

1. Determine all of the subgroups of $\mathbb{Z}_{12}$ under addition modulo 12. Justify your answer.

2. Let n be a natural number. Make a conjecture by finishing this sentence: All of the subgroups of $\mathbb{Z}_n$ are of the form ...

3. Prove that the set $\mathbb{R}^2$ is a group under vector addition.

4. Find a subgroup of $\mathbb{R}^2$. Then find another.

5. Let $G = \{2^n | n \in \mathbb{Z}\}$. Show that G is a subgroup of $\mathbb{R}^*$, the nonzero real numbers under multiplication.

6. We've seen that $GL_2(\mathbb{R})$, the set of 2×2 matrices with entries from $\mathbb{R}$ and nonzero determinants, forms a group under matrix multiplication. Prove that $SL_2(\mathbb{R}) = \{M \in GL_2(\mathbb{R}) | det(M) = 1\}$ is a subgroup of $GL_2(\mathbb{R})$.
   NOTE: Recall that for two square matrices A and B of the same size, $det(AB) = det(A)det(B)$.

7. We can form matrices with entries from $\mathbb{C}$ and use the familiar matrix operations.

   Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $J = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, and $K = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

   Show that the set $H = \{\pm E, \pm I, \pm J, \pm K\}$ is a group under matrix multiplication by making its multiplication table and explaining how the axioms hold. (You may assume associativity.) Is this group abelian?

8. Find all of the subgroups of H (from the above problem).