Math 302 – Abstract Algebra

**Problem Set 2**

Due Thursday, March 3

1. Prove that the set $T = \{z \in \mathbb{C} : |z| = 1\}$ is a group under multiplication. (You may take as given that multiplication in the complex numbers is associative.)

2. Referring to Problem 1 above, notice that in the complex plane, $T$ is the circle of radius 1 centered at the origin. For $z \in T$, define the *argument* of $z$, written $arg(z)$, to be the angle $\theta$ (in radians), $0 \leq \theta < 2\pi$, such that $z = \cos\theta + i\sin\theta$. So, for example, $arg(i) = \pi/2$. Prove that for $z, w \in T$, $arg(zw) = arg(z) + arg(w)$.
NOTE: you may use these trigonometric identities:

   - $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
   - $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

3. Prove that the set $U = \{z \in \mathbb{C} : z^6 = 1\}$ is a group. Then list the elements of $U$ in standard form (that is, the form $a + bi$, where $a$ and $b$ are real numbers), and plot them in the plane.

4. Let $D_n$ denote the group of symmetries of a regular $n$-sided polygon, where $n > 2$ is an integer, and $\rho_0$ is the identity element. ($D_n$ is called a *dihedral* group.) Suppose that $\mu_1$ and $\mu_2$ are distinct reflections in $D_n$. Prove that $\mu_1\mu_2 \neq \rho_0$.

5. Give an example of a group $G$ and elements $a$ and $b$ in $G$ such that $a^{-1}ba \neq b$.

6. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group?

7. Prove that every group table is a *Latin Square*; that is, each element of the group appears exactly once in each row and in each column. (This is what we've been calling the Sudoku Property.) NOTE: An infinite group has a group table that extends down and to the right forever; be sure to cover the infinite case in your proof.

8. Prove that if $G$ is a group with the property that the square of every element is the identity, then $G$ is abelian.