Math 302 – Abstract Algebra

Problem Set 1

Due Thursday, February 25

1. Consider an infinitely long strip of equally spaced H's:
   ... H H H H H ...
   Describe the symmetries of this strip. Is the composition operation commutative? Explain.

2. Recall from Worksheet 2 that the set of complex numbers $\mathbb{C}$ is $\{a + bi \mid a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$.

   (a) Show that $\mathbb{C}$ is *closed* under addition; that is, show that the sum of two complex numbers is also a complex number. What is the additive identity in $\mathbb{C}$?

   (b) Show that $\mathbb{C}$ is closed under multiplication. What is the multiplicative identity in $\mathbb{C}$?

   (c) Let $z = a + bi$ be any complex number. The *modulus* of z is the real number $|z| = \sqrt{a^2 + b^2}$. (Note that this is a different use of the word "modulus" than the number-theoretic one; context is important.) Show that for any two complex numbers z and w, $|zw| = |z||w|$.

   (d) Given a nonzero complex number $z = a + bi$, find its multiplicative inverse and write it the form $\alpha + \beta i$, where α and β are real numbers (in terms of a and b).

   (e) By establishing a correspondence between the complex number $z = a + bi$ and the ordered pair (a, b), we describe the *complex plane* in order to visualize $\mathbb{C}$. Notice that in this setup, $|z|$ is the length of the vector from the origin to (a, b). What is the relationship in the plane between a nonzero complex number z and its multiplicative inverse? (Try several examples.)

3. Using the First Principle of Mathematical Induction, prove that for any natural number n, $1 + 3 + 5 + ... + (2n - 1) = n^2$. That is,
   (i) Show that the equation is true when $n = 1$, and
   (ii) Assume that the equation is true when $n = k$, and show that the equation is true for $n = k + 1$.

4. For any positive integer n, prove that $2^n 3^{2n} - 1$ is always divisible by 17.

5. Let $A = 1, 2, 3, ..., 10, 11$, and make a multiplication table modulo 12. Find a set U contained in A such that a multiplication table modulo 12 using only the elements in U has what we've been calling the "Sudoku Property." (So in the body of your new smaller table, every element of U shows up exactly once in each row and column.) How would you describe the elements of U? (If you've had Number Theory, use language that a classmate who hasn't had that course would understand.)