Math 302 Worksheet 13

Consider the Cayley tables for the dihedral groups D_3 and D_4 . You may have noticed that in each case, the rotations form a subgroup, and they occupy the top left and bottom right quadrants of the chart. The reflections (actually the only other coset of this subgroup) occupy the other quadrants. If you squint, you might see a table that looks like the Cayley table for $\mathbb{Z}_2$. In this worksheet, we formalize and generalize this idea. As it turns out, the subgroup in question needs a special property.

Definition. A subgroup H of a group G is a *normal* subgroup of G if $aH = Ha$ for all $a \in G$.

Notice that the definition concerns an equality of sets.

1. Using the definition, show that $< \rho_1 >$ is a normal subgroup of D_3.
2. Prove the **Normal Subgroup Test**: A subgroup H is normal in G if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.
3. Let $H = \{(1), (12)\}$. Show that H is not normal in S_3.
4. Let G be a group and let H be a normal subgroup of G, and define the set $G/H = \{aH | a \in G\}$, the set of cosets of H in G. Prove that G/H is a group under the operation $(aH)(bH) = abH$.
   NOTE: Start by showing that the operation is well-defined. That is, show that if $aH = a'H$ and $bH = b'H$ for some $a, a', b, b' \in G$, then $abH = a'b'H$ (so you get the same product of two cosets regardless of which coset representatives you use to compute that product). You should be making clear use of the definition of normal subgroup.
5. Actually you've worked with factor groups already! For example, $\mathbb{Z}_5$ is isomorphic to G/H for what group G and what subgroup H of G?
6. The *index* of a subgroup H in a group G is the number of left cosets of H in G. Prove that if H has index 2 in G, then H is normal in G.