Math 302 Worksheet 10

You may have noticed a pair of groups which seem to be structurally the same, even if they aren't identical.

Definition. An *isomorphism* ϕ from a group G to a group $\bar{G}$ is a one-to-one mapping from G onto $\bar{G}$ that preserves the group operation, meaning that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.
If there is an isomorphism from G onto $\bar{G}$, we say that G and $\bar{G}$ are *isomorphic* and write $G \approx \bar{G}$.

1. Let G be a cyclic group of order 12 with generator a. Prove that G is isomorphic to $\mathbb{Z}_{12}$. To do this,
    a) Define a function from G to $\mathbb{Z}_{12}$.
    b) Prove that ϕ is one-to-one by assuming that $\phi(a) = \phi(b)$ and proving that $a = b$.
    c) Prove that ϕ is onto; that is, for any $\bar{g} \in \mathbb{Z}_{12}$, find an element $g \in G$ such that $\phi(g) = \bar{g}$.
    d) Prove that ϕ is operation-preserving; that is, show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.
2. Find an isomorphism from the group of integers under addition to the group of even integers under addition.
3. Show that $U(8)$ is isomorphic to $U(12)$. Is $U(8)$ isomorphic to $U(10)$? If not, find a group that is isomorphic to $U(10)$.
4. Let ϕ be an isomorphism from a group G to a group $\bar{G}$. Prove
    a) ϕ carries the identity of G to the identity of $\bar{G}$.
    b) For every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$.
    c) $|a| = |\phi(a)|$ for all $a \in G$.
5. Prove that isomorphism is a transitive relation on the set of all groups. That is, show that if G, H, and K are groups such that $G \approx H$ and $H \approx K$, then $G \approx K$.
6. Prove that S_4 is not isomorphic to D_{12}. (HINT: Consider orders of elements.)