Abstract Algebra
Problem Set 8

The Fundamental Theorem of Finite Abelian Groups. Every finite Abelian group is a direct of cyclic groups of prime-power order. More specifically, every such group is isomorphic to a group of the form $\mathbb{Z}_{{p_1}^{n_1}} \times \mathbb{Z}_{{p_2}^{n_2}} \times \cdots \times \mathbb{Z}_{{p_k}^{n_k}}$, where the primes p_i are not necessarily distinct, and the prime powers are uniquely determined by G.

The Fundamental Theorem allows us to determine isomorphism classes of finite Abelian groups. For example, we can see that any Abelian group of order 4 (actually there are no non-Abelian groups of order 4, but that's another story) is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. So up to isomorphism, there are two Abelian groups of order 4.

1. Find a subgroup of D_4 that is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
2. Up to isomorphism, how many groups of order 16 are there? List them as direct products of groups of the form $\mathbb{Z}_n$, as above.
3. For each of the groups in the previous problem, how many elements of order 2 are there? How many of order 4?
4. Prove that any Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9?
5. How many Abelian groups (up to isomorphism) are there
    a) of order 6?
    b) of order 15?
    c) of order 42?
    d) of order pq, where p and q are distinct primes?
    e) of order pqr, where p, q, and r, are distinct primes?
    f) Generalize parts d and e.
6. Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.
7. Characterize those integers n such that the only Abelian groups of order n are cyclic.
8. Characterize those integers n such that any Abelian group of order n belongs to one of exactly four isomorphism classes.