Abstract Algebra
Problem Set 6

Definition. Let a and b be integers. We write $gcd(a, b)$ to denote the greatest common divisor of a and b.

Definition. Let $n > 1$ be an integer. The set
$U(n) = \{k \in \mathbb{Z} : 0 < k < n, gcd(n, k) = 1\}$.

1. Prove that for any integer $n > 1$, $U(n)$ forms a group under multiplication mod n.

2. Is $U(n)$ always cyclic, sometimes cyclic, or never cyclic? Justify your answer.

3. Prove that the set $\{\sigma \in S_4 : \sigma(2) = 2\}$ is a subgroup of S_4.

4. List all of the subgroups of S_4. Identify which of them are also subgroups of A_4.

5. Prove that S_n is nonabelian for $n \geq 3$.

6. Prove that A_n is nonabelian for $n \geq 4$.

7. Consider a regular tetrahedron (four sides, each one an equilateral triangle). Imagine labeling the corners, and then using that labeling to keep track of the symmetries of the tetrahedron. Which subgroup of S_4 corresponds to the group of all symmetries of the tetrahedron? Explain.

8. Let G be a group, and let $g \in G$. Define the map $\lambda_g : G \to G$ by $\lambda_g(a) = ga$. Prove that λ_g is a permutation of G.