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Even and Odd Permutations

Recall that any permutation can be written as a product of transpositions (2-cycles), and that there are many ways to do so. For example, note that $(12345) = (54)(53)(52)(51) = (54)(52)(21)(25)(23)(13)$.

Definition. A permutation is *even* if it can be expressed as the product of an even number of transpositions. A permutation is *odd* if it can be expressed as the product of an odd number of transpositions.

The purpose of this handout is to demonstrate that the definition of evenness or oddness does not depend on a particular representation of a permutation as a product of transpositions, and hence that the above definition makes sense.

Lemma. Suppose that $\sigma_1, \sigma_2, ...\sigma_n$ are transpositions in S_n, and that $\sigma_1\sigma_2 \cdot ... \cdot \sigma_{n-1}\sigma_n = \epsilon$, the identity permutation. Then n is even.

Proof: Without loss of generality, we may assume that the rightmost transposition $\sigma_n = (12)$. We will focus on the number 1, and notice that there are three possibilities for σ_{n-1}:

1. $\sigma_{n-1} = (12)$
2. $\sigma_{n-1} = (ab)$, where neither a nor b is 1 or 2.
3. $\sigma_{n-1} = (1c)$, where $c \neq 1$, $c \neq 2$.

In the first case, $\sigma_{n-1}\sigma = \epsilon$, so we can write $\sigma_1\sigma_2 \cdot ... \cdot \sigma_{n-3}\sigma_{n-2} = \epsilon$.

In the second case, because σ_{n-1} and σ_n are disjoint, they commute, and we can write $\sigma_1\sigma_2 \cdot ... \cdot \sigma_{n-2}(12)(ab) = \epsilon$.

In the third case, $\sigma_{n-1}\sigma_n = (1c)(12) = (12c) = (12)(2c)$ (check this!), so we can write $\sigma_1\sigma_2 \cdot ... \cdot \sigma_{n-2}(12)(2c) = \epsilon$.

Notice that we've either eliminated exactly two transpositions, or we have a product in which the final transposition has no 1 in it. Continuing in this fashion, we either get complete cancellation of all transpostions, in which case we know that n is even, or we reach the point where we have $\tau_1\tau_2 \cdot ... \cdot \tau_k = \epsilon$, where the τ's are transpositions, k is even, and 1 does not appear in τ_2 or any transposition to the right of τ_2. In the second case, 1 cannot appear in τ_1 either, because if it did, the product could not equal ϵ, since the permutation would not fix 1.

We can now repeat the entire procedure by focusing on d, say, where $\tau_k = (df)$ for some numbers d and f, eliminating transpositions two at a time. Eventually we'll get down to none, proving that n must be even. $\square$

Theorem. Let $\alpha \in S_n$. Then either any expression of α as a product of transpositions has an even number of transpositions, or any such expression has an odd number of transpositions.

Proof: Suppose $\alpha = \beta_1\beta_2\cdots\beta_j = \gamma_1\gamma_2\cdots\gamma_k$,
where the β's and γ's are all transpositions. Recalling that a transposition is its own inverse, we see that
$\beta_1\beta_2\cdots\beta_j\gamma_k^{-1}\cdots\gamma_2^{-1}\gamma_1^{-1} = \epsilon$, the identity permutation. By the Lemma above, $j+k$ is an even number. Therefore either j and k are both even or they are both odd. $\square$