Name: SOLUTIONS
PROBLEM SET Due Date: F 10/29 4pm

Topics/Chapters: Magnetic fields, magnetic force, Biot-Savart Law, Ampère's Law
Relevant Class Meetings: 10/20 W + 10/22 F + 10/25 M

1. Griffiths 5.12 – two line charges, \( \vec{B} \) balances \( \vec{E} \)
2. \( \vec{B} \)-field from rectangular frame – superposition of straight wires; uses Griffiths Ex. 5.5
3. Griffiths 5.9 – choose (a) or (b): field for curved wire configurations
4. Solenoid – Biot-Savart, superposition
5. Current-carrying wire – \( \vec{B} \)-field inside and outside of wire

1. **Griffiths 5.12.** Two line charges, \( \vec{B} \) balances \( \vec{E} \).

Collaborators:

Two key concepts that you used:

Useful pages in book and/or dates from class:

Note: when solving this problem, see footnotes 4 and 8 in Griffiths' Ch. 5.

Problem 5.12 Suppose you have two infinite straight line charges \( \lambda \), a distance \( d \) apart, moving along at a constant speed \( v \) (Fig. 5.26). How great would \( v \) have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number... Is this a reasonable sort of speed?\(^6\)

\(^6\)If you've studied special relativity, you may be tempted to look for complexities in this problem that are not really there—\( \lambda \) and \( v \) are both measured in the laboratory frame, and this is ordinary electrostatics (see footnote 4).

\[ \vec{B}_1 = \frac{\mu_0 I \hat{\phi}}{2\pi s} \]  
(Using results from Griffiths, pg 217)

![Figure 5.26](image-url)
The force on the current \( I_2 \) is found using

\[
\vec{F}_{\text{mag}} = I \int \vec{dl} \times \vec{B}, \quad \text{from} \quad \vec{F} = \int (\vec{v} \times \vec{B}) \, dq = \int (\vec{v} \times \vec{B}) \, \vec{dl}
\]

\[
\Rightarrow \quad \vec{F}_{\text{mag}} = I_2 \int \vec{dl} \times \vec{B}, \quad \text{here, with} \quad I_2 = \lambda v.
\]

We know \( \vec{dl}_2 \parallel \hat{z} \) and \( \vec{B}_1 \parallel \hat{r} \Rightarrow |\vec{dl}_2 \times \vec{B}_1| = |\vec{dl}'| |\vec{B}|
\]

\[
\Rightarrow \quad F_{\text{mag}} = I_2 B_1 |\vec{dl}'| \quad \text{(and this force is attractive)}
\]

So if we just consider a segment of length \( L \),

\[
F_{\text{mag}} = I_2 B_1 L
\]

or force per unit length

\[
f_{\text{mag}} = \frac{F_{\text{mag}}}{L} = I_2 B_1,
\]

\[
\begin{align*}
I_2 &= \lambda v \\
B_1 &= \frac{\mu_0 I}{2 \pi d} = \frac{\mu_0 \lambda v}{2 \pi d}
\end{align*}
\]

\[
\Rightarrow \quad f_{\text{mag}} = \frac{\mu_0 (\lambda v)^2}{2 \pi d}
\]

**Electric Force**

Simple electrostatics can be used here, since the charge density is constant in time \( [\text{footnotes 8, 4}] \).

Wire 1 produces an \( \vec{E} \)-field, which creates a force on \( 2 \)

\[
\vec{E}_1 = \frac{\lambda_1}{2 \pi \varepsilon_0} \hat{S}
\]

(can be found from Gauss's law)

The force on the charge density \( \lambda_2 \) is found by considering the force on an infinitesimal part \( dq \).
\[\frac{dF_{\text{elec}}}{d\lambda_2} = \dot{E}_1 \varepsilon_0\varepsilon_0 \quad (\text{from } F = q\varepsilon_0 E)\]

\[F_{\text{elec}} = \int \varepsilon_1 \varepsilon_0 \lambda_2 \text{d}l\]

So we can consider a segment of length \(L\)

\[F_{\text{elec}} = E_1 \lambda_2 L \quad (\text{which is a repulsive force})\]

or force per unit length

\[\dot{F}_{\text{elec}} = \frac{F_{\text{elec}}}{L} = E_1 \lambda_2\]

\[
\begin{align*}
\lambda_2 &= \lambda \\
E_1 &= \frac{\lambda}{2\pi\varepsilon_0 d}
\end{align*}
\]

\[\Rightarrow \quad \dot{F}_{\text{elec}} = \frac{\lambda^2}{2\pi\varepsilon_0 d}\]

Now invoke the condition \(F_{\text{mag}} = F_{\text{elec}}\)

\[\Rightarrow \quad \frac{\mu_0 \lambda^2 v^2}{2\pi d} = \frac{\lambda^2}{2\pi\varepsilon_0 d}\]

Solve for speed \(v\):

\[v = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \quad \text{I happen to know this is the speed of light...}\]

...but I can check using

\[\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2\]

\[\varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2\]

\[v = \sqrt{\frac{10.19}{4\pi \times 8.85}} \Rightarrow \quad v \approx 3.0 \times 10^8 \text{ m/s}\]

Since this is the speed of light, it is not a reasonable sort of speed.

For realistic speeds, \(E\)-force will be dominant.
2. \( \mathbf{B} \)-field for a rectangular frame. Superposition of fields from straight wires.

Collaborators: 

Two key concepts that you used: 

Useful pages in book and/or dates from class: 

Note: when solving this problem, use a method similar to Griffiths' Ex. 5.5 to find the field from each straight segment of wire.

Current flows in the wire frames shown above.

(a) By explicit calculation of the field from each segment of wire in the left-hand figure, find the magnetic field (direction and magnitude) at the point \( P \) at the center of the cube.

**These results from Griffiths' are useful:**

![Figure 5.18](image)

![Figure 5.19](image)

He finds the field at point \( P_{\text{griffiths}} \) to be

\[
\mathbf{B}(P_{\text{griffiths}}) = \frac{\mu_0 I}{4\pi r} (\sin \theta_2 - \sin \theta_1)
\]

pointing out of page.

Use this result to find fields for wire segments in cubic configurations above.
In the frame on the left, each segment can be analyzed individually, using the following geometry:

\[ P = (0,0,0) \]
\[ \sqrt{2b^2} = b\sqrt{2} \]
\[ \sqrt{3b^2} = b\sqrt{3} \]

View in the plane formed by \( P \) and a segment of wire.

\[ b \]
\[ \theta \]
\[ +\theta \]

View from a side face of cube:

\[ 2b \]
\[ \sqrt{2b^2} \]

This point is located a distance \( b \) in the horizontal direction and \( b \) in the vertical direction away from \( P \) (hence the length \( b\sqrt{2} \) for the separation).

Use \[ B = \frac{\mu_0 I}{4\pi S} (\sin b_2 - \sin \theta_1) \]

We can apply our formula with \( \theta_1 = -\theta, \theta_2 = \theta \)

And \[ \sin \theta = \frac{b}{b\sqrt{3}} \Rightarrow \sin \theta = \frac{1}{\sqrt{3}} \]

We know \( \sin(-\theta) = -\sin(\theta) \)

\[ \Rightarrow \sin \theta_2 - \sin \theta_1 = -\sin \theta_1 + \sin \theta = 2 \sin \theta \]

Substitute \( S = b\sqrt{2} \) and expression for \( \sin \theta \):

\[ \text{Call this } B_0 = \frac{\mu_0 I}{4\pi (b\sqrt{2})} \left( 2 \frac{1}{\sqrt{3}} \right) \Rightarrow B_0 = \frac{\mu_0 I}{2\pi b\sqrt{6}} \text{ for one segment.} \]

Griffiths has shown (or started) this in Exercise 5.5.

Here's another look @ it:

The field will point in the "\( \hat{\phi} \)" direction for each segment.

\[ \hat{\mathbf{B}} = \int d\mathbf{B} = \frac{\mu_0}{4\pi} \int \hat{\mathbf{a}} \times \mathbf{dl}' \]

\( P \)
\( \theta \)
\( \theta' \)

This is the origin and observation point:

\( \mathbf{r} = 0 \]
\( \mathbf{r}' = -\mathbf{r} \)

\( \mathbf{r}_a = (s', \mathbf{0}, -2) \]
\( \mathbf{r}_b = (s, \mathbf{0}, -2) \)

\( \pi \)

\[ \hat{\mathbf{a}} \]
\[ \hat{\mathbf{b}} \]

Need to find \( \hat{\mathbf{a}} \times \hat{\mathbf{b}} \) direction.
To help me find \( \hat{n} \times \hat{n} \) directions, I will write \( \hat{n} \) vectors

\[
\hat{a}_a = \ell_s \hat{s} + \ell_q \hat{q} + \ell_{a_z} \hat{a}_z
\]
\[
\hat{a}_b = \ell_s \hat{s} + \ell_q \hat{q} + \ell_{b_z} \hat{a}_z
\]

where I've used \( \ell a_q = \ell b_q = \ell q = 0 \); \( \ell a_s = \ell b_s = \ell s = \ell \) for this geometry.

Also, I know \( \ell a_z = -\ell b_z = -z' \)

\[
\hat{a}_a = s \hat{s} - z' \hat{a}_z
\]
\[
\hat{a}_b = s \hat{s} + z' \hat{a}_z
\]

**Explicit \( \hat{n} \times \hat{n} \)**

\[\hat{a}_a \times \hat{a}_b = \left( I \hat{a} \right) \times \frac{1}{\ell} \left( s \hat{s} - z' \hat{a}_z \right)\]
\[= \frac{I}{\ell} \left( s \hat{s} \times \hat{s} - z' \hat{a}_z \times \hat{a}_z \right) = \frac{I s}{\ell} \hat{q} \]

\[\hat{b} \times \hat{b} = \left( I \hat{b} \right) \times \frac{1}{\ell} \left( s \hat{s} + z' \hat{a}_z \right)\]
\[= \frac{I}{\ell} \left( s \hat{s} \times \hat{s} + z' \hat{a}_z \times \hat{a}_z \right) = \frac{I s}{\ell} \hat{q} \]

**Explicit \( \hat{B} \)**

So for one segment of wire,

\[dB(\rho) = \frac{\mu_0}{4\pi} \left( \frac{\hat{a}_a \times \hat{a}_a}{\ell_a^2} dl_a + \frac{\hat{b} \times \hat{a}_b}{\ell_b^2} dl_b \right)\]

where \(-b \leq l'_a \leq 0\) and \(-l \leq b \leq +b\) \(\ell_a = \ell_b = \ell\)

\[\hat{B} = \int dB = \int_0^b \frac{2\mu_0}{4\pi} \left[ \frac{1}{\ell^2} \left( \frac{I s}{\ell} \right) dl' \hat{q} \right] \text{ for one segment.}\]
THE INTEGRAL CAN BE EVALUATED OUT:

\[ \mathbf{b} = \frac{\mu_0 I s}{2 \pi} \int_0^b \frac{1}{\sqrt{r^2 + (z')^2}^{3/2}} \, dz' \hat{z} \]

Use \( \int \frac{1}{r^2} \, dx = \frac{1}{a^2} x + C \) if \( r = (a^2 + x^2)^{1/2} \)

\[ \Rightarrow \mathbf{b} = \frac{\mu_0 I s}{2 \pi} \left[ \frac{2}{s^3 (s^2 + (z')^2)^{3/2}} \right]_0^b \hat{z} \]

Use \( b = \sqrt{s^2 + b^2} = \sin \theta \)

\[ \Rightarrow \mathbf{b} = \frac{\mu_0 I}{2 \pi s^2} \sin \theta \hat{z} \]

AS WE EXPECT.

ALL THIS IS JUST TO SHOW THE DIRECTION OF \( \mathbf{b} \) FOLLOWS

RIGHT-HAND RULE WITH NO EXTRA COMPONENTS.

IT'S FINE TO SKIP THIS AND SAY SIMPLY THAT WE KNOW THE DIRECTION OF \( \mathbf{b} \).

SINCE THE GEOMETRY IS A CUBE, WE KNOW THAT

THE FIELD FROM EACH SEGMENT WILL BE AT SOME

45° ANGLE RELATIVE TO THE X, Y, Z COORDINATES IN

THE FIRST (LEFT-HANDED) FIGURE.

FROM THIS POINT, I WILL SHOW TWO METHODS TO FIND THE FIELD.

1. **Explicitly find** \( \mathbf{b} \) **for each of**
   
   THE 8 SEGMENTS **AND ADD ALL FIELDS**

2. **Argue that fields from 4 segments cancel, and then find \( \mathbf{b} \) explicitly**
   
   FOR THE REMAINING 4 SEGMENTS
METHOD #1

Use Griffiths' results directly as given, and then deal with vector addition at the end.

Find the field from each of the 8 segments:

1. Look in \(-\hat{y}\) direction, at the cross-section

   \[ \vec{B}_1 = \frac{B_0}{\sqrt{2}} (-\hat{x} + \hat{z}) \]
   \[ \vec{B}_7 = \frac{B_0}{\sqrt{2}} (\hat{x} + \hat{z}) \]

   [Already we can see these will cancel!]

2. Look in \(-\hat{z}\) direction (down from the top)

   \[ \vec{B}_2 = \frac{B_0}{\sqrt{2}} (-\hat{x} + \hat{y}) \]
   \[ \vec{B}_6 = \frac{B_0}{\sqrt{2}} (\hat{x} + \hat{y}) \]

3. Look in \(-\hat{x}\) direction (from front)

   \[ \vec{B}_4 = \frac{B_0}{\sqrt{2}} (\hat{y} - \hat{z}) \]
   \[ \vec{B}_8 = \frac{B_0}{\sqrt{2}} (\hat{y} + \hat{z}) \]
We find the total field by adding all these contributions and using our expression for $B_0$.

$$B_0 = \left( \frac{\mu_0 I}{2\pi b \sqrt{3}} \right)$$

**TOTAL FIELD:**

$$\sum_{i=1}^{8} B_i$$

adding components from all 8 segments!

$$= \frac{B_0}{\sqrt{2}} \left( -6\hat{x} + 6\hat{x} + 4\hat{y} + 6\hat{z} - 6\hat{z} \right) = \frac{4}{\sqrt{2}} B_0 \hat{y}$$

$$\Rightarrow \quad B_{\text{TOTAL}} (P) = \frac{4}{\sqrt{2}} \left( \frac{\mu_0 I}{2\pi b \sqrt{3}} \right) \hat{y} = \frac{4\mu_0 I}{2\sqrt{2}(12) \pi b \sqrt{3}} \hat{y}$$

**METHOD #2**

**MAKE ARGUMENTS OF SYMMETRY INITIALLY, AND THEN APPLY GRIFFITHS' RESULTS**

**SYMMETRY CONSIDERATION:** EXAMINE FIELD AT P DUE TO WIRES $\theta$, $\theta$, $\theta$, $\theta$

**STATEMENT #1:** $\hat{y}$ FIELD DUE TO EACH SEGMENT WILL ONLY HAVE COMPONENT IN $\hat{y}$ DIRECTION

[IF YOU DESIRE JUSTIFICATION, SEE THE BEGINNING OF THIS SOLUTION]
STATEMENT # 2 : \( B \)-FIELDS DUE TO \( 1, 3, 5, 7 \) CANCEL

JUSTIFY : EXAMINE THE CROSS-SECTION VIEW

\[ B_1 \quad \bullet \quad - \quad B_5 \]

\[ B_3 \quad \bullet \quad - \quad B_7 \]

\( \text{THESE VECTORS ADD TO ZERO} \)

\[ B_3 \quad \bullet \quad - \quad B_7 \]

NOW THAT WE HAVE DEVELOPED THIS SIMPLIFICATION DUE TO SYMMETRY, WE CAN PROCEED TO CALCULATE THE SUPERPOSITION OF FIELDS FROM THE OTHER FOUR WIRES!

WE FOUND (IN METHOD #1) THAT

\[ B_2 = \frac{B_0}{\sqrt{2}} (-\hat{x} + \hat{y}) \]

\[ B_4 = \frac{B_0}{\sqrt{2}} (\hat{y} - \hat{x}) \]

\[ B_6 = \frac{B_0}{\sqrt{2}} (\hat{x} + \hat{y}) \]

\[ B_8 = \frac{B_0}{\sqrt{2}} (\hat{y} + \hat{x}) \]

WITH \( B_0 = \frac{\mu_0 I}{2 \pi b \sqrt{6}} \)

WHICH YIELDS \( \text{\( \vec{B}_{\text{TOTAL}} = \vec{B}_2 + \vec{B}_4 + \vec{B}_6 + \vec{B}_8 \)} \)

\[ \Rightarrow \vec{B}_{\text{TOTAL}} = \frac{B_0}{\sqrt{2}} \left( (-\hat{x} + \hat{y}) + 4\hat{y} + (-\hat{x} + \hat{y}) \right) = \frac{4B_0}{\sqrt{2}} \hat{y} \]

\[ = \frac{4}{\sqrt{2}} \left( \frac{\mu_0 I}{2 \pi b \sqrt{6}} \right) \hat{y} = \left( \frac{4}{12 \sqrt{2}} \right) \left( \frac{\mu_0 I}{2 \pi b \sqrt{6}} \right) \hat{y} \]

\[ \Rightarrow \vec{B}_{\text{TOTAL}} = \frac{\mu_0 I}{\sqrt{3} \pi b} \hat{y} \]

AS WE FOUND WITH METHOD #1
(b) Show that the field at P is the same as if the wire frame on the left were replaced by the single square loop on the right. Why is this true?

The single loop (right-hand figure) can be analyzed using the same technique as in (a).

1. \( \vec{B}_1 = \frac{B_0}{\sqrt{2}} \left( \hat{y} + \hat{z} \right) \)
2. \( \vec{B}_3 = \frac{B_0}{\sqrt{2}} \left( \hat{y} - \hat{z} \right) \)
3. \( \vec{B}_2 = \frac{B_0}{\sqrt{2}} (x + \hat{y}) \)
4. \( \vec{B}_4 = \frac{B_0}{\sqrt{2}} (-x + \hat{y}) \)

And so the total field will be

\[
\vec{B}_{\text{tot}} = \sum_{i=1}^{4} \vec{B}_i = \frac{B_0}{\sqrt{2}} \left( (x - \hat{x}) + (\hat{z} - \hat{z}) + 4\hat{y} \right)
\]

\[
\Rightarrow \quad \vec{B}_{\text{tot}} = 4 \frac{B_0}{\sqrt{2}} \hat{y}
\]

And so \( \vec{B}_{\text{tot}} = \frac{\mu_0 I}{\sqrt{3} \pi b} \hat{y} \) as we found in (a).

By superposition, we see that the total field is the same as (a).

Note that \( \Theta_b \) and \( \Theta_a \) are identical to \( \Theta_a \) and \( \Theta_b \).

And that \( \Theta_b \) and \( \Theta_b \) have the same fields at P as \( \Theta_a \) and \( \Theta_a \), respectively. For a pair of wires, the field is the same on the "RHS" and "LHS":

- \( \vec{E} \) into plane

\( f_1 \quad f_2 \) if field is the same at P, and \( f_2 \) is the same as \( f_1 \)

- \( \vec{E} \) out of plane
3. Griffiths 5.9. Choose (a) or (b). Field for curved wire configurations.

Collaborators: 

Two key concepts that you used: 

Useful pages in book and/or dates from class: 

Problem 5.9 Find the magnetic field at point $P$ for each of the steady current configurations shown in Fig. 5.23.

(a) 
(b) 

Figure 5.23

We know that the field at $P$ will have no contribution from the straight-line segments. Why? We nearly see $\vec{B} \parallel \vec{I}$, and also, we saw that the field calculated at Point $P$ near a straight-line segment is given by

\[ B = \frac{\mu_0 I}{4\pi s} (\sin \theta_2 - \sin \theta_1) \]

So if $P$ is on an axis of current, $s \to 0$ but also $\theta_1 = \theta_2 \Rightarrow \vec{B} \to 0$

See this because $\sin \theta_1 = \frac{\Delta x_1}{\sqrt{\Delta x_1^2 + s^2}}$, $\sin \theta_2 = \frac{\Delta x_2}{\sqrt{s^2 + \Delta x_2^2}}$.

\[ \sin \theta_1 = \frac{1}{\sqrt{1 + \frac{s^2}{\Delta x_1^2}}} \approx 1 - \frac{1}{2} \left( \frac{s^2}{\Delta x_1^2} \right) + \cdots \quad \text{(Binomial expansion)} \]

\[ \sin \theta_2 = \frac{1}{\sqrt{1 + \frac{s^2}{\Delta x_2^2}}} \approx 1 - \frac{1}{2} \left( \frac{s^2}{\Delta x_2^2} \right) + \cdots \]

\[ \Rightarrow \sin \theta_2 - \sin \theta_1 \approx -\frac{1}{2} s^2 \left( \frac{1}{\Delta x_2^2} - \frac{1}{\Delta x_1^2} \right) \]

And $B \approx \frac{\mu_0 I}{8\pi} s \left( \frac{1}{\Delta x_1^2} - \frac{1}{\Delta x_2^2} \right)$ so $B \to 0$ as $s \to 0$. 

The contribution to the field due to the curved portions can be calculated by the method used in Example 5.10 and in class on 10/22, based on Biot-Savart Law.

The field will only have a component out of the page (right-hand rule) and the value will be

\[ \mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{r^2} \]

We can say that the origin and observation points are the same

\[ \Rightarrow \quad \mathbf{r} = 0, \quad \mathbf{r}' = -\mathbf{r}' \quad \text{and} \quad \mathbf{r}^2 = \text{constant} \]

We also see \( dl' = s \, d\phi' \), integrated between limits \( \phi_1 \) and \( \phi_2 \).

And in this problem, \( \phi_1 = 0 \), \( \phi_2 = \frac{\pi}{2} \).

For each arc we have:

\[ |\mathbf{B}_a| = B_a = \frac{\mu_0 I}{4\pi} \int_{\phi_1}^{\phi_2} \frac{a \, d\phi'}{a^2} = \frac{\mu_0 I}{4\pi a} \left( \frac{\pi}{2} - 0 \right) \]

\[ \Rightarrow B_a = \frac{\mu_0 I}{8a} \]

Similarly, \( |\mathbf{B}_b| = B_b = \frac{\mu_0 I}{8b} \)

By the right-hand rule, \( \mathbf{B}_a \parallel z \) and \( \mathbf{B}_b \parallel -\mathbf{z} \).

\[ \Rightarrow \quad \mathbf{B}(P) = \frac{1}{8} \mu_0 I \left( \frac{1}{a} - \frac{1}{b} \right) \hat{z} \]

(out of page)
The straight segments can be analyzed using Griffith's Ex. 5.5 results:

\[ B_{\text{straight}} = \frac{M_o I}{4\pi s} (\sin \theta_2 - \sin \theta_1) \]

And for one segment spanning 0 to +\(\infty\), \(\theta_1 = \frac{\pi}{2}\)

\[ B_0 = B_{0, +\infty} = \frac{M_o I}{4\pi s} (1 - 0) = \frac{M_o I}{4\pi s} \]

Right-hand rule tells us that the field due to 0 will point into the paper and the field due to 2 will also:

\[ \Rightarrow B_0 + B_2 = 2 \left( \frac{M_o I}{4\pi s} \right) \]

\[ = \frac{M_o I}{2\pi s} \text{ into page} \]

\[ \therefore \overrightarrow{\hat{B}} \cdot \overrightarrow{OP} \]

The curved segment can be analyzed using results of Ex. 5.6:

\[ B_{\text{curve}} = \frac{M_o I}{4\pi s} \int dq' \quad \text{(as in (a))} \]

With \(q_1 = 0\), \(q_2 = \pi\) here.

\[ \Rightarrow \overrightarrow{B}_2 = \frac{M_o I}{4\pi s} (\pi - 0) = \frac{M_o I}{4\pi} \]

And right-hand rule (\(\vec{\hat{B}} \times \hat{\mathbf{n}}\)) tells us \(\hat{\mathbf{B}}\) points into the page.

Combine all these:

\[ \overrightarrow{B}_{\text{total}} = \overrightarrow{B}_0 + \overrightarrow{B}_2 + \overrightarrow{B}_3 \]

\[ \overrightarrow{B}_{\text{total}} = \frac{M_o I}{4\pi s} (2 + \pi) \text{ into the page} \]
4. **Solenoids.** Biot-Savart Law and fields for finite coils, infinite coils, and sheets.

Collaborators:

Two key concepts that you used:

Useful pages in book and/or dates from class:

Note: this problem has several parts, the first of which is the most difficult. You answers for the second and third parts might be quite short.

**(a) Griffiths 5.11.** Use Biot-Savart Law to find field on the axis of a solenoid. Here are figures that might be useful to you, from Purcell (left two figures) and from Griffiths (right). Express you answer for a finite solenoid in terms of $\theta_1$ and $\theta_2$, and then find the field for an infinite solenoid. Note that the Griffiths figure shows point P outside of the solenoid, while Purcell’s figure uses a point inside the coil.

![Solenoid diagrams](image)

**(b) For a finite solenoid, does it matter if point P is inside the solenoid (Purcell figure) or outside the solenoid (Griffiths)?**

---

To calculate the field at P, use two concepts:

**CONCEPT #1:** Field on axis of a ring of current

\[
B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}
\]

**CONCEPT #2:** Solenoid is like a stack of rings, so we can find field due to superposition of stacked rings (each with field $B_1(z)$)
IMPLEMENT THESE IDEAS.

A PORTION OF THE SOLENOID WITHIN dθ HAS THICKNESS dz

TOTAL CURRENT IN THIS PORTION IS \( dI = I_0 n dz \)

\[
\begin{align*}
B_{\text{TOTAL}} &= \int dB = \int \left[ \frac{\mu_0 I_0 n dz}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \right] \\
&= \frac{\mu_0 I_0 n}{2} \int_{z_1}^{z_2} \frac{a^2}{(a^2 + z^2)^{3/2}} dz \\
&= \frac{\mu_0 I_0 n a^2}{2} \left[ \frac{\frac{z}{\sqrt{a^2 + z^2}}}{\left( \frac{a}{\sqrt{a^2 + z^2}} \right)^{3/2}} \right]_{z_1}^{z_2} \\
&= \frac{\mu_0 I_0 n a^2}{2} \left[ \frac{\frac{z_2}{\sqrt{a^2 + z_2^2}}}{\left( \frac{a}{\sqrt{a^2 + z_2^2}} \right)^{3/2}} - \frac{\frac{z_1}{\sqrt{a^2 + z_1^2}}}{\left( \frac{a}{\sqrt{a^2 + z_1^2}} \right)^{3/2}} \right] \\
&= \frac{\mu_0 I_0 n a^2}{2} \left[ \frac{z_2}{(a^2 + z_2^2)^{1/2}} - \frac{z_1}{(a^2 + z_1^2)^{1/2}} \right]
\end{align*}
\]

WRITE USING \( \Theta_1, \Theta_2 \ldots \)
From geometry,

\[ \cos \theta_1 = \frac{z_1}{\sqrt{a^2 + z_1^2}} ; \quad \cos \theta_2 = \frac{z_2}{\sqrt{a^2 + z_2^2}} \]

Direction is known from field for one ring Z

\[ B = \frac{\mu_0 I_0}{2} \left( \cos \theta_2 - \cos \theta_1 \right) \]

Where \( I = I_0 = \text{current in wire of solenoid} \), \( B \) direction comes from right-hand rule

And for an infinite solenoid, \( \theta_1 = -\pi \), \( \theta_2 = 0 \)

\[ B = \frac{\mu_0 I_0}{2} \left( 1 - (-1) \right) \quad \Rightarrow \quad B = \frac{\mu_0 I_0}{2} \]

**Method #2**

Write in terms of angle and integrate \( d\theta \)

\[ dB = \frac{\mu_0 I_0}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \, dz \]

Now I use Griffith's geometry, to assign angles:

(Note! \( \theta \) could be just an angle, not the angle in spherical coordinates \( r, \theta, \phi \) or cylindrical coordinates \( s, \phi, z \) — it's just there in the geometry of this problem.)

![Diagram](image-url)
We can write \( \sin \theta = \frac{\pi \, d \theta}{dz} \) \( \Rightarrow \) \( dz = \frac{\pi \, d \theta}{\sin \theta} \)

\[ dB = \frac{\mu_0 I_0 n}{2} \left( \frac{a^2}{a^2 + z^2} \right)^{3/2} \left( \frac{\pi \, d \theta}{\sin \theta} \right) \]

AND KEEP GOING...

\[ = \frac{\mu_0 I_0 n}{2} \left( \frac{a^2}{\lambda^2} \right) \left( \frac{\pi \, d \theta}{\sin \theta} \right) = \frac{\mu_0 I_0 n}{2} \left( \frac{a^2}{\lambda^2} \right) \left( \frac{d \theta}{\sin \theta} \right) \]

But \( \sin \theta = \frac{a}{\lambda} \)

\[ dB = \frac{\mu_0 I_0 n}{2} \left( \sin^2 \theta \right) \left( \frac{d \theta}{\sin \theta} \right) \]

GOOD!

\[ B = \frac{\mu_0 I_0 n}{2} \int \sin \theta \, d \theta \]

\[ \text{Hence } B \text{ is measured away from vertical axis,} \]
\[ \text{so the integral along the solenoid will} \]
\[ \text{run from } \theta_{\text{bottom}} \text{ to } \theta_{\text{top}} \text{ (as } d \theta \text{ increases)} \]

\[ \text{or, in Griffith's notation,} \]
\[ \text{from } \theta_2 \text{ to } \theta_1 \text{.} \]

\[ \text{Alternatively, I could integrate} \]
\[ \text{in the } -d \theta \text{ direction from} \]
\[ \theta_1 = \theta_{\text{top}} \text{ to } \theta_2 = \theta_{\text{bottom}} \]
\[ \text{(as } d \theta \text{ decreases)} \]

\[ B = \frac{\mu_0 I_0 n}{2} \int_{\theta_2}^{\theta_1} \sin \theta \, d \theta = \frac{\mu_0 I_0 n}{2} \int_{\theta_1}^{\theta_2} \sin \theta (-d \theta) \]

\[ \text{and this yields} \]

\[ B = \frac{\mu_0 I_0 n}{2} \left( -\cos \theta_1 - (-\cos \theta_2) \right) \]

\[ \text{so} \]

\[ B = \frac{\mu_0 I_0 n}{2} \left( \cos \theta_2 - \cos \theta_1 \right) \]

\[ \text{as previously seen!} \]
Note: This can also be done as a surface integral with but savant, from scratch:

\[ B = \frac{\mu_0}{4\pi} \int \frac{r \times \hat{n}}{r^2} \, da' \]

with \( da' = s\, dp' \, dz' \)

A given length \( L \) contains \( N \) turns, each with current \( I_0 \), so the surface current density is

\[ K = (I_0)(N)(\frac{1}{L}) \]

Here I will use the expression \( \hat{I}_0 \, n \) as the surface current density:

\[ \hat{K} = \frac{d\hat{I}_0}{dl} = (\hat{I}_0 \, n) \frac{1}{L} = \hat{I}_0 \, n \]

\[ B = \left[ \frac{\mu_0}{4\pi} \int \frac{\hat{I}_0 \times \hat{n}}{\sqrt{z^2}} \, n \, da' \right] \hat{z} + \left[ \frac{\mu_0}{4\pi} \int \frac{\hat{I}_0 \times \hat{n}}{L^2} \, n \, da' \right] \hat{K} \]

(there is no \( \hat{\varphi} \) component)

The calculation can be carried out from here over the double integral \( dp', dz' \).

Evaluate \( \hat{I}_0 \times \hat{n} \) ...

\[ \hat{I}_0 = I_0 \hat{\varphi} \]

\[ \hat{n} = s \hat{z} + a \hat{q} + \varphi \hat{r} \]

From this point, use conversion to write vectors in terms of \( x, y, z \).

Then find components \( |\hat{I}_0 \times \hat{n}| \, x, y, z \)

And evaluate integrals. I will not show these steps here,

but that is the general idea!
NOTE: It's fairly obvious from looking at the results that the limit of an infinite solenoid is evaluated easily when \( \vec{B} \) is written in terms of \( \theta_1 \) and \( \theta_2 \). It is more difficult to evaluate \( \vec{B} \) in terms of \( z_1 \to -\infty \) and \( z_2 \to +\infty \).

(b) It certainly does not matter if \( P \) is inside or outside, since we do the calculation for the finite solenoid with \( P \) drawn outside but then let it be inside for the infinite case. Since we develop this analysis from the superposition of individual rings, there is no limit on the position of our observation point \( P \), provided it is along the axis of the solenoid.
(c) The sheet of current shown in the figure below has uniform surface current density \( K \) Coulombs/cm. What is the strength of the magnetic field along the central axis, in terms of \( K \)? Base your analysis on your answer from part (a).

The surface current density can be identified by going back to this idea:

\[
dI_1 = n I \, dz
\]

This single ring is just a small portion of a sheet, so we use the definition of \( K \)

\[
\Rightarrow |K| = K = \frac{dI_1}{dz} = n I
\]

And in our expression for \( B \) we substitute \( K = nI \)

\[
\Rightarrow B = \mu_0 K (\cos \theta_2 - \cos \theta_1) \quad \text{(finite)}
\]

\[
B = \mu_0 K \quad \text{(infinite)}
\]
Problem 5.13 A steady current \( I \) flows down a long cylindrical wire of radius \( a \) (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if
(a) The current is uniformly distributed over the outside surface of the wire.
(b) The current is distributed in such a way that \( J \) is proportional to \( z \), the distance from the axis.

(a) Griffiths 5.13(a).

(b) Griffiths 5.13(b).

(c) Find the magnetic field both inside and outside the wire, if the volume current density \( J \) is uniform throughout the material.

**Use Ampère’s Law:** 
\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}
\]

**Note:** Since the wire has cylindrical symmetry, \( \mathbf{B} \) only has component in \( \hat{\phi} \) direction, and \( d\mathbf{l} \parallel \hat{\phi} \)

\[
\Rightarrow \mathbf{B} \cdot d\mathbf{l} = B dl = 2\pi s B
\]

\[
\begin{align*}
\text{Inside:} & \quad I_{\text{enc}} = 0 \quad \Rightarrow \quad B = 0 \\
\text{Outside:} & \quad I_{\text{enc}} = I \quad \Rightarrow \quad 2\pi s B = \mu_0 I \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi s}
\end{align*}
\]

\[
B = \begin{cases} 
0 & \text{inside} \\
\frac{\mu_0 I}{2\pi s} & \text{outside}
\end{cases}
\]
Let \( J = \alpha s \)

\[ d\alpha = sdsd\psi \frac{2}{3} ; \quad \frac{T}{\|d\alpha} \]

Inside:

\[ I_{\text{enc}} = \int_{0}^{s} J(s') \cdot d\alpha \]

\[ = \int_{0}^{s} (\alpha s) (sdsd\psi) = 2\pi \alpha s \frac{s^3}{3} \]

We can write \( \alpha \) in terms of total current \( I \):

\[ I = \int_{0}^{a} J \cdot d\alpha = \frac{2\pi \alpha a^3}{3} \implies \alpha = \frac{3I}{2\pi a^3} \]

And so inside the wire, \( I_{\text{enc}} = \left( \frac{2\pi s^3}{3} \right) \left( \frac{3I}{2\pi a^3} \right) = \frac{I s^3}{a^3} \)

\[ \implies 2\pi s B = \mu_0 \left( \frac{I s^3}{a^3} \right) \implies B = \frac{\mu_0 I s^2}{2\pi a^3} \]

Outside: Same as in part (a)

\[ B = \left\{ \begin{array}{ll}
\frac{\mu_0 I s^2}{2\pi a^3} & \text{inside} \\
\frac{\mu_0 I}{2\pi s} & \text{outside}
\end{array} \right. \]

Here \( J = J_0 = \text{constant} \)

\[ \vec{J} \cdot d\vec{\alpha} \]

Inside:

\[ I_{\text{enc}} = \int_{0}^{s} J_0 (sdsd\psi) = J_0 (2\pi) \left( \frac{1}{2} s^2 \right) = J_0 \pi s^2 \]

Find \( I = \int_{0}^{a} J_0 (sdsd\psi) \implies J_0 = \frac{I}{\pi a^2} \)

So inside:

\[ 2\pi s B = \mu_0 \left( \frac{I s^2}{a^2} \right) \implies B = \frac{\mu_0 I s}{2\pi a^2} \]

Outside: Same as before!

\[ B = \left\{ \begin{array}{ll}
\frac{\mu_0 I s}{2\pi a^2} & \text{inside} \\
\frac{\mu_0 I}{2\pi s} & \text{outside}
\end{array} \right. \]
(d) In a few sentences, how do you account for uniform current distribution inside a wire?

Griffiths writes, "It may have occurred to you that since parallel wires attract, the current within a single wire should contract into a tiny concentrated stream along the axis. Yet in practice the current typically distributes itself quite uniformly over the wire." [Griffiths, pg. 247]

Purcell agrees: "Since parallel current filaments attract one another, one might think that a current flowing in a solid rod ... would tend to concentrate near the axis of the rod. That is, the conduction electrons, instead of distributing themselves evenly as usual over the interior of the metal, would crowd in toward the axis and most of the current would be there." [Purcell pg. 249]

What prevents this 'concentration' from happening?

* The electric field between the charged particles prevents them from becoming concentrated at the center of the wire. As we saw in problem #1 (Griffiths 5.12), the electric force will be significantly larger than the magnetic force unless the charges move reasonably fast. The magnetic force pulling the moving charges together will be offset by the repulsive electronic force.

* When we think of real wires, there will be zero electronic force if the positive and negative charges are uniformly distributed. When current flows, we can imagine that the electronic force will increase as the magnetic force wants to concentrate the moving charges. In this way the two forces counteract each other and there is no concentration of charges even in neutral wires.