Math 200 Fall 2015
November 2

**Definition.** Let $A$ be an $m \times n$ matrix. The *transpose* of $A$, denoted $A^T$, is the $n \times m$ matrix obtained from $A$ by writing the rows of $A$, in order, as columns. So for $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$, $A^T = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} \\ a_{12} & a_{22} & \ldots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix}$.

**Example.** For $A = \begin{bmatrix} 1 & 6 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 & 0 \\ 6 & -1 & 4 \end{bmatrix}$.

**Theorem.** If $A$ and $B$ are matrices of the same dimensions, then $(A + B)^T = A^T + B^T$.

The next theorem is included for your use in the homework. To make sense of the proof, you might want to work through an example or two.

**Theorem.** If $A$ is an $m$ by $k$ matrix, and $B$ is a $k$ by $n$ matrix (so the product $AB$ is defined), then $(AB)^T = B^T A^T$.

**Proof:** The row $i$, column $j$ entry of $AB$ is the dot product of row $i$ of $A$ with column $j$ of $B$ (it’s $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$). This becomes the row $j$, column $i$ entry of $(AB)^T$. On the other hand, the corresponding row $j$, column $i$ entry of $B^T A^T$ is the dot product of row $j$ of $B^T$ with column $i$ of $A^T$, which gives the same result. $\square$

**HOMEWORK DUE WEDNESDAY, NOVEMBER 4:**

1. Explain why $(A^T)^T = A$ for any matrix $A$.

2. **Definition.** A square matrix $A$ is *symmetric* if $A = A^T$.
   
   a) Give examples of $2 \times 2$, $3 \times 3$, and $4 \times 4$ symmetric matrices.
   
   b) Prove: If $B$ is any square matrix, then $B + B^T$ is symmetric. (Confirm with a few examples before you prove in general.)

   c) Prove: If $C$ is any square matrix, then $CC^T$ is symmetric.
3. **Definition.** A square matrix \( A \) is *skew-symmetric* if \( A = -A^T \).

a) Give examples of \( 2 \times 2 \), \( 3 \times 3 \), and \( 4 \times 4 \) skew-symmetric matrices.

b) Show that in any skew-symmetric matrix, the entries on the main diagonal \((a_{ii}, i = 1, ..., n)\) are all zeros.

c) Prove: If \( B \) is any square matrix, then \( B - B^T \) is skew-symmetric. (Again, you might want to confirm with a few examples first.)

4. Show that any square matrix \( C \) is the sum of a symmetric matrix and a skew-symmetric matrix.

5. Show that if a square matrix \( A \) is symmetric, then so is \( A^2 = AA \).