Math 200 Spring 2015
February 27

Notice that in $\mathbb{R}^2$, if $\vec{v}$ is a scalar multiple of $\vec{w}$, say $\vec{v} = k \vec{w}$, where $k \neq 0$, then we can rearrange the equation and write $(1)\vec{v} + (-k)\vec{w} = \vec{0}$. In words, we can write the zero vector as a linear combination of $\vec{v}$ and $\vec{w}$, where the coefficients are not zero. It is the second equation we will use as a model to define linear dependence and independence in $\mathbb{R}^3$.

**Definition.** The vectors $\vec{v}_1, \vec{v}_2, ... \vec{v}_n \in \mathbb{R}^3$ are **linearly dependent** if there are scalars $a_1, a_2, ... a_n$, not all zero, with $a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}$.

In other words, a set of vectors is linearly dependent if there is a nontrivial (meaning scalar coefficients are not all 0) linear combination of them equal to the zero vector.

**Example:** $\vec{v}_1 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 & 0 & -3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 & -4 & -5 \end{bmatrix}$ are linearly dependent, because $4\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$. (You worked with this example on the last homework.)

**Definition.** The vectors $\vec{v}_1, \vec{v}_2, ... \vec{v}_n \in \mathbb{R}^3$ are **linearly independent** if the only scalars $a_1, a_2, ... a_n$ such that $a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}$ are $a_1 = 0, a_2 = 0, ...$, and $a_n = 0$.

**Example.** The vectors $\vec{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, and $\vec{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ are linearly independent. To see this, note that the only way to have $a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \vec{0}$ is with $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$; that is, $a_1 = a_2 = a_3 = 0$.

**Definition.** A **basis** for $\mathbb{R}^3$ is a set $\mathcal{B}$ of vectors that both span $\mathbb{R}^3$ and are linearly independent.

**Example.** The set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the **standard basis** for $\mathbb{R}^3$.

**HOMEWORK DUE MONDAY, MARCH 2:**

1. Show that the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent. That is, show that the only solution to $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$ is the (trivial) solution $a = 0, b = 0, c = 0$.

   NOTE: In the homework due March 5, you showed that these vectors span $\mathbb{R}^3$. Thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$. 
2. For each of the following sets of vectors $\mathbf{B} = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$, determine whether or not $\mathbf{B}$ is a basis for $\mathbb{R}^3$. To do this, consider the vector equation $a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = \mathbf{w}$, where, say, $\mathbf{w} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$. This equation corresponds to a system of 3 equations in the variables $a_1, a_2, a_3$. (i) Does the system have a solution for all possible values of $p, q,$ and $r$? If so, then $\mathbf{B}$ spans $\mathbb{R}^3$. (ii) For the special case of $\mathbf{w} = \mathbf{0}$, is the only solution the one with $a_1 = 0, a_2 = 0,$ and $a_3 = 0$? If so, then $\mathbf{B}$ is a linearly independent set. If $\mathbf{B}$ both spans $\mathbb{R}^3$ and is linearly independent, then it forms a basis for $\mathbb{R}^3$.

a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$

b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$

c) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

3. Recall that the set of solutions to $x + 2y + 3z = 0$ forms a plane through the origin in $\mathbb{R}^3$.

a) Show that the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ lies on this plane.

b) Show that the vector $\mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ also lies on this plane.

c) Let $j$ and $k$ be two scalars. Show that the vector $\mathbf{u} = j\mathbf{v} + k\mathbf{w}$ also lies in the plane.

4. GIVEN: If the dot product of two vectors in $\mathbb{R}^3$ equals zero, then the vectors are perpendicular.

a) Find a vector $\mathbf{v}$ that is perpendicular to $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

b) Find a vector $\mathbf{w}$ that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

5. Find a vector that is perpendicular to all vectors lying in the plane $x + 2y + 3z = 0$. 