What is Numerical Linear Algebra?

Definition

the creation, study, and implementation of algorithms to solve problems from linear algebra

- creation - understand and address challenges, propose a simplification or modification
- study - discover error estimates, predict performance
- implementation - code, run, examine, adapt
What is Numerical Linear Algebra?

**Definition**

the creation, study, and implementation of algorithms to solve problems from linear algebra

- **solve** - find approximate ("good enough") or exact solutions
- **problems** - linear systems ($A\vec{x} = \vec{b}$), eigenvalue problems, ill-posed problems
Problem

Determine the amount of traffic between each of the four intersections.
Mass Balance

\[(\text{rate in}) - (\text{rate out}) = 0\]

\[
\begin{align*}
(w + x) - (100 + 200) &= 0, \\
(150 + 100) - (x + y) &= 0, \\
(y + z) - (125 + 75) &= 0, \\
(100 + 150) - (w + z) &= 0
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
w \\
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
300 \\
250 \\
200 \\
250
\end{bmatrix}
\]
This system has infinitely many solutions! To uniquely determine traffic flow, one must include an additional observation point.

Given \( A\vec{x} = \vec{b} \)
- There is a unique solution.
- There are infinitely many solutions.
- There are no solutions.
Well-posed and Ill-posed Problems

Well-posed
a unique solution exists

For a linear system:

\[ A\vec{x} = \vec{b} \]

if \( A \) is a square nonsingular matrix (\( A^{-1} \) exists), then \( A\vec{x} = \vec{b} \) is a well-posed problem.

Ill-posed
- there is more than one solution (possibly infinitely many)
- there is no solution
- the solution is unstable (small changes in data produce large changes in the solution)
Today’s Linear Problems

- often VERY large – not solvable by hand!
- involve messy real data – making accurate calculations difficult
- usually part of a larger problem
- no such thing as a ‘one-size-fits-all’ solution method

The need for Numerical Linear Algebra

We need **practical methods** to find “good enough” solutions for today’s problems!
Given a $100 \times 100$ matrix, $A$, of random numbers and a $100 \times 1$ vector, $b$, of random numbers, solve $Ax = b$.

- $\text{inv}(A)$-computes the inverse of $A$, then performs $x = A^{-1}b$
- $A\backslash b$- solves $Ax = b$ without computing inverse
- GE-row reduce $[A|b]$ and backsolve

<table>
<thead>
<tr>
<th></th>
<th>inv(A)</th>
<th>$A\backslash b$</th>
<th>GE</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>0.015866</td>
<td>0.00088432</td>
<td>0.017442</td>
</tr>
<tr>
<td>error</td>
<td>$1.9991e-13$</td>
<td>$1.5739e-13$</td>
<td>$1.7252e-13$</td>
</tr>
</tbody>
</table>

Not all methods are equivalent!!!
Given a $1000 \times 1000$ matrix, $A$, of random numbers and a $1000 \times 1$ vector, $b$, of random numbers, solve $Ax = b$.

<table>
<thead>
<tr>
<th></th>
<th>$\text{inv}(A)$</th>
<th>$A \backslash b$</th>
<th>GE</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>0.035627</td>
<td>0.017455</td>
<td>1.7936</td>
</tr>
<tr>
<td>$10 \times \text{size}$</td>
<td>$2.25 \times \text{longer}$</td>
<td>$380 \times \text{longer}$</td>
<td>$1950 \times \text{longer}$</td>
</tr>
<tr>
<td>error</td>
<td>$4.0745e-11$</td>
<td>$8.122e-12$</td>
<td>$1.5859e-11$</td>
</tr>
</tbody>
</table>

*Comp. cost increase (time/storage) > dim. increase*
Given a $10,000 \times 10,000$ matrix, $A$, of random numbers and a $10,000 \times 1$ vector, $b$, of random numbers, solve $Ax = b$.

<table>
<thead>
<tr>
<th></th>
<th>inv(A)</th>
<th>A\ b</th>
<th>GE</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>13.512</td>
<td>5.4065</td>
<td>3509.6</td>
</tr>
<tr>
<td>10×size</td>
<td>319×longer</td>
<td>310×longer</td>
<td>1960×longer</td>
</tr>
<tr>
<td>error</td>
<td>1.5813e−09</td>
<td>3.8528e−10</td>
<td>3.7676e−10</td>
</tr>
</tbody>
</table>

*Time is money!!! Efficiency matters!!!*
MATH0328 Learning Goals

Our goals are to

- study the motivation, creation, implementation, and analysis of numerical methods in linear algebra
- develop mathematical tools/skills for matrix analysis
- understand the challenges and role of numerical computing in mathematics today
<table>
<thead>
<tr>
<th><strong>Prerequisites</strong></th>
<th>MATH0200 (Linear Algebra)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text</strong></td>
<td><em>Numerical Linear Algebra</em> by Layton and Sussman</td>
</tr>
<tr>
<td></td>
<td>ISBN: 978-1-312-32985-0</td>
</tr>
<tr>
<td></td>
<td>pdf: <a href="http://www.lulu.com/spotlight/Layton_Sussman">http://www.lulu.com/spotlight/Layton_Sussman</a></td>
</tr>
<tr>
<td><strong>Resources</strong></td>
<td>Wordpress Site:</td>
</tr>
<tr>
<td></td>
<td><a href="http://sites.middlebury.edu/math0328/">http://sites.middlebury.edu/math0328/</a></td>
</tr>
<tr>
<td><strong>Software</strong></td>
<td>We will use MATLAB.</td>
</tr>
<tr>
<td></td>
<td>No prior experience programming is necessary.</td>
</tr>
<tr>
<td></td>
<td>We will learn some basic coding.</td>
</tr>
<tr>
<td></td>
<td>Advanced coding is entirely optional.</td>
</tr>
<tr>
<td><strong>Readings</strong></td>
<td>Readings outlined in weekly announcements on site.</td>
</tr>
<tr>
<td></td>
<td>Students expected to read prior to class.</td>
</tr>
</tbody>
</table>
Grades  
40% weekly homework  
40% Exams (2 written exams during semester)  
20% Project (weeks 6-end of semester)

Homework  
- weekly homework assignments  
- mix of hand calculations, proofs, and MATLAB  
- working together is encouraged  
- writing alone is required  
- not meant to be completed in one sitting!

Exams  
- 2 in-semineter exams  
- take-home, open note  
- no working together

Project  
- completed in pairs  
- choose your own topic  
- begin (approximately) week 6  
- formal paper and presentation during finals
Expectations

- **Be here:** Attend all lectures, arriving on time, and staying for the duration of the class period.

- **Be prepared:** Complete assigned readings and homework problems prior to attending class. Do not stress about understanding every detail you read, but focus on getting a general picture of the topics, and understanding some of the examples.

- **Be present:** Plan to participate by both asking and answering questions, as well as by taking part in discussions and group activities.

- **Be proactive** in your understanding. Start assignments early. Ask questions as they come to you. Attend office hours for clarification the moment you run into trouble.

- **Be respectful** of yourself, your classmates, your professor, and our classroom. We are all responsible for ensuring a successful semester as a productive, welcoming, inclusive, and stimulating class environment.