MATH0328: Numerical Linear Algebra
Homework 6

Due Wednesday, April 12

Instructions

Research in Numerical Linear Algebra can take many forms. One type of research in this area is algorithm development, analysis, and testing (conducting numerical experiments). New algorithms may be developed from scratch, or adapted from existing algorithms. This homework mimics what a research project in Numerical Linear Algebra would look like.

Recall the First-Order Richardson Method:

Algorithm 1 (FOR).

\textit{Given} $A \in \mathbb{R}^{n \times n}$ \textit{and} $b \in \mathbb{R}^n$, \textit{choose} $\rho > 0$. \textit{Guess} $x_0$. \textit{Compute the next iterates by the iteration process given below:}

$$\rho(x^{n+1} - x^n) = b - Ax^n.$$  

The mathematical motivation behind the development of this algorithm is that $Ax = b$ can be written as a root-finding equation: $0 = b - Ax$. The algorithm is created out of the identity $\rho(x - x) = b - Ax$ for any $\rho > 0$. In our analysis of the method we concluded that when $A$ is SPD, FOR converges when $\rho > \frac{\lambda_{\max}}{2}$, and converges in the fewest iterations when $\rho = \rho_{opt} = \frac{\lambda_{\min} + \lambda_{\max}}{2}$.

Suppose we want to use this algorithm to design a new and (hopefully) improved variation of FOR. Our thought process may be: if one step of FOR in each iteration gets us closer to the true solution ($\|e^1\| \leq \|T\|\|e^0\|$, and if $\|T\| < 1$, we are closer than our previous guess), what if we perform two steps of FOR in each iteration? Will this get us twice as close in each iteration? Would such a method then converge in fewer iterations? We wish to explore this idea.
It begins with a mathematical formulation of the idea above (creation of the algorithm). We define 2-step FOR as follows.

**Algorithm 2 (2-Step FOR).**

Given $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^{n}$, choose $\rho_1, \rho_2 > 0$. Guess $x_0$. Compute the next iterates by the 2-step iteration process:

- $\rho_1(x_{n+1/2} - x^n) = b - Ax^n$,
- $\rho_2(x^{n+1} - x^{n+1/2}) = b - Ax^{n+1/2}$.

\[ X^{n+1} = \begin{array}{c} \circ \end{array} + \begin{array}{c} \sim \end{array} \mathcal{X}^n \]
Analysis

The following problems outline the steps that you, as a researcher, would perform to complete a theoretical convergence analysis of 2-step FOR.

1. Eliminate the half-step ($x^{n+1/2}$) so that 2-step FOR can be written as a single step involving the iterates $x^n$ and $x^{n+1}$.

2. Recall that in regular FOR, $x^{n+1} = \frac{1}{\rho}b + Tx^n$ where the iteration matrix, $T$ is given by $T = I - \frac{1}{\rho}A$. Show that 2-step FOR can be written in a similar form. That is, show that

$$x^{n+1} = \tilde{b} + \tilde{T}x^n,$$

for some $\tilde{b}$ and iteration matrix $\tilde{T} = T_1T_2$, with $T_i = I - \frac{1}{\rho_i}A$.

3. Prove that $e^{n+1} = \tilde{T}e^n$. (The proof will be similar to what we did when proving the same result for FOR). Proving this allows us to use that for any matrix norm $\|\cdot\|$, $\|e^n\| \leq \|\tilde{T}\|^n\|e^0\|$, which is critical for our convergence analysis.

4. Find (with proof) the eigenvalues of the iteration matrix $\tilde{T}$ in terms of the eigenvalues of $A$.

5. Determine sufficient restriction(s) on choice of $\rho_1$, $\rho_2$ so that 2-step FOR converges when $A$ is SPD.
   (You must find when $\|\tilde{T}\| \approx \text{spr}(\tilde{T}) < 1$.)

6. (BONUS 5 pts. to be awarded to Exam 2 total) Find $\rho_1^{opt}$ and $\rho_2^{opt}$ when $A$ is SPD. Use our analysis of 1-step FOR as your guide.

7. Predict whether or not 2-step FOR will be an improvement over 1-step FOR when $A$ is SPD. Specifically, predict which method will converge within a specified tolerance in the fewest iterations. Give mathematical (not numerical) justification for your prediction based on your convergence analysis.
Numerical Experiments

Numerical experiments are an important part of any algorithm development and analysis. In research, numerical experiments are often taking place during the theoretical research process. This is because “theory inspires practice, and practice inspires theory.” Numerical experiments can inform a researcher whether or not their algorithm is effective, as well as give ideas for how the algorithm may be improved, or what type of convergence is expected (knowing what you want to prove is a big advantage). At the same time, theoretical analysis may draw attention to more effective ways to program and implement an algorithm, improving its practice.

1. Create a MATLAB code for 2-step FOR applied to the 1D MPP problem from HW5, problem 6c. You can use the code from that homework as a template to start your file. Test your function file when \( n = 4, \ h = 1/5, \) and \( f(x) = 5^3x \) using \( \rho_{1,2} = 2 \). Check that you obtain the following iterates (answers below were rounded to two decimal places):

\[
\begin{align*}
\mathbf{u}^1 &= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2.75 \end{bmatrix}, \\
\mathbf{u}^{10} &= \begin{bmatrix} 3.93 \\ 6.89 \\ 7.89 \\ 5.93 \end{bmatrix}, \\
\mathbf{u}^{20} &= \begin{bmatrix} 4 \\ 6 \\ 7 \\ 5 \end{bmatrix}
\end{align*}
\]

2. (BONUS 3 pts. added to HW total) How many iterations are needed for 2-step FOR with \( \rho_{1,2} = 2 \), to converge within a tolerance of \( 10^{-8} \) when \( n = 4, \ h = 1/5, \) and \( f(x) = 5^3x \)? Is this an improvement over 1-step FOR? Does this agree with your theoretical analysis?

3. (BONUS 3 pts. added to HW total) Repeat the previous experiment for (at least) the following choices of \( \rho_{1,2} \) with \( \text{itMax} \) set at \( 2 \times 10^5 \). You may want to test more choices of \( \rho_{1,2} \). Discuss whether or not the results agree with your analysis of the method.

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<thead>
<tr>
<th>Test</th>
<th>( \rho_1 )</th>
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<tr>
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<td>3</td>
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4. (BONUS 3 pts. added to HW total) Having completed the analysis and the above numerical experiments, what are your ideas to continue your research? Discuss and outline your plans.