MATH0328: Numerical Linear Algebra
Homework 3

Due Wednesday, March 8

Instructions

Complete the following problems. Show all work. Only use MATLAB on
problems that are marked with (MATLAB). Include a published pdf of your
MATLAB script. There is a template m-file available on the course website
for problems 7-9. Solutions to other problems should be written by hand or
typeset in \LaTeX.

Problems

1. (text p.51 Exercise 113) Prove that if $\langle \cdot, \cdot \rangle_*$ is an inner product, then
   \[ \|\vec{x}\|_* = \sqrt{\langle \cdot, \cdot \rangle_*} \] is a norm.  (Hint: You are proving Proposition 108.
   Using definition 107, verify that the three properties of Definition 103
   hold.)

2. (text p.51 Exercise 116) The unit ball is $\{ \vec{x} : \|\vec{x}\|_* \leq 1 \}$. Sketch
   the unit ball in $\mathbb{R}^2$ for the $1$, $2$ and $\infty$ norms. Note that the only
   ball that looks ball-like is the one for the $2$–norm. Sketch the unit
   ball in the weighted $2$–norm induced by the inner product $\langle x, y \rangle :=
   (1/4)x_1y_1 + (1/9)x_2y_2$.

3. (text p.56 Exercise 124) Calculate the $1$, $2$, and $\infty$ norms of each
   matrix (do not use MATLAB, complete by hand):

   $$A_1 = \begin{bmatrix} 1 & -3 \\ -4 & 7 \end{bmatrix},$$
   $$A_2 = \begin{bmatrix} 1 & -3 \\ -3 & 7 \end{bmatrix}.$$
4. (text p.56 exercise 125) Show that an orthogonal change of variables preserves the $2 -$norm. In other words, prove the statement:

$$\text{If } O^T O = I \text{ then } \|O\vec{x}\|_2 = \|\vec{x}\|_2.$$ 

5. (text p.56 Exercise 126) Prove that, for an \textit{induced} matrix norm,

$$\|I\|_* = 1, \text{ and } \|A^{-1}\|_* \geq 1/\|A\|_*,$$

where $<\cdot, \cdot>_*$ is an inner product (so $\|A\|_* = \max_{\vec{x} \in \mathbb{R}^N \setminus \{\vec{0}\}} \frac{\|A\vec{x}\|_*}{\|\vec{x}\|_*}$).

6. (text p.59 Exercise 131) If $\|\cdot\|$ is a vector norm and $U$ an $N \times N$ nonsingular matrix, show that $\|\vec{x}\|_* := \|U\vec{x}\|$ is a vector norm (\textit{Show that $\|\vec{x}\|_*$ satisfies the three properties of Definition 103}). When $\|\cdot\| = \|\cdot\|_2$, find a formula for the matrix norm induced by $\|\cdot\|_*$. 

7. (MATLAB) Complete the following.

(a) Use the matlab commands \texttt{max}, \texttt{sum}, \texttt{sqrt}, and \texttt{abs} to calculate the $\infty$-norm, 1-norm, and 2-norm of the following vectors. You may find \texttt{.}-arithmetic helpful: \textit{Array vs. Matrix Operations}.

(b) Verify that your calculations are correct by using the \texttt{MATLAB} command \texttt{norm}. Display your results in a table.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ [-17] \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1e-17 \\ 200 \\ -2 \\ 1e4 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} \pi \\ \pi/2 \\ \pi/4 \\ \pi/8 \\ \pi/16 \end{bmatrix}$$
8. (MATLAB) Complete the following (without using a for-loop).

(a) (MATLAB) Use the matlab commands \texttt{max}, \texttt{sum}, and \texttt{abs} to calculate the \(\infty\)-norm, 1-norm, and Frobenius norm of the matrix \(A\) given below. (Complete without using a \texttt{for-loop}. You may find .-arithmetic helpful: Array vs. Matrix Operations.

(b) Verify that your calculations are correct by using the MATLAB command \texttt{norm}. Display your results in a table.

\[
A = \begin{bmatrix}
17 & -24 & 1 & -8 & 15 \\
-24 & 5 & -7 & -14 & 15 \\
4 & -1 & 13 & 1 & 22 \\
10 & 12 & 19 & -21 & 3 \\
11 & -10 & 25 & 2 & 9 \\
\end{bmatrix}
\]

9. (MATLAB) Complete the following (without using a for-loop).

(a) Calculate the eigenvalues of the matrix \(A\) in the previous problem using the MATLAB command \texttt{eig}. (Type \texttt{help eig} for more on how to use this command.)

(b) Calculate the singular values of \(A\) (these are the square roots of the eigenvalues of \(A^T A\)).

(c) Verify that the 2-norm of \(A\), given in MATLAB by \texttt{norm(A)}, is the square root of the largest eigenvalue of \(A^T A\). Notice that it does not equal the Frobenius norm of \(A\).